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# The Helicity of the Velocity Field for Cellular Convection in a Rotating Layer

**A. V. Getling***

*Skobeltsyn Institute of Nuclear Physics, Lomonosov Moscow State University, Moscow, Russia*

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**Abstract**—The helicity of a cellular convective flow in a horizontal layer of a compressible fluid (gas) heated from below and rotating about the vertical axis is studied using finite-difference numerical simulations. The medium is assumed to be polytropically stratified. A thermal perturbation that produces a system of Bénard-type hexagonal convection cells is introduced at the initial time. Next, the cells are deformed by the action of the Coriolis force; however, at some stage of the evolution, the flow is nearly steady (at later times, the cells break down). For given Rayleigh and Prandtl numbers, the velocity-field helicity for this stage averaged over the layer increases with decreasing polytrope index (i.e., with increasing the curvature of the static entropy profile) and has a maximum at a certain rotational velocity of the layer. Numerical simulations of such quasi-ordered convective flows should reduce the uncertainties in estimates of the helicity, a quantity important for the operation of MHD dynamos.

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## 1. INTRODUCTION

After the discovery of the generation of large-scale magnetic fields by turbulent conductive fluid motions [1, 2], the focus of investigations of the origin and evolution of the global solar magnetic field moved rapidly to mean-field electrodynamics and the theory of turbulent dynamos based on it. As a result, studies of generation mechanisms based on the action of turbulence long ago became predominant in this area of solar physics [3].

The effect of a turbulent flow on the magnetic field is usually considered in a kinematic approach, without taking into account the back action. Let us assume that the turbulence is homogeneous and weakly anisotropic. Then, for the mean emf vector

$$\mathcal{E} \equiv \langle [\mathbf{u} \times \mathbf{b}] \rangle, \tag{1}$$

due to the interaction between the velocity pulsations $\mathbf{u}$ and the magnetic-field pulsations $\mathbf{b}$,[1] a quasi-linear approximation (first-order smoothing frequently applied in mean-field electrodynamics [4, 5]) in the lowest-order terms of the expansion in the small quantity reciprocal of the spatial scale of the variations in the mean field $\mathbf{B}$ yields

$$\mathcal{E} = \alpha\mathbf{B} - \beta[\nabla \times \mathbf{B}]. \tag{2}$$

(The traditional notation $\alpha$ for the coefficient of proportionality in the first term of this expression gave rise to the term that has come to be generally accepted to denote the generation of the mean emf, the $\alpha$ *effect*.)

In view of (2), the mean-field induction equation for the turbulent medium can be written

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \{[\mathbf{U} \times \mathbf{B}] + \alpha\mathbf{B} - [\nabla \times (\nu_{\mathrm{m}} + \beta)\mathbf{B}]\}, \tag{3}$$

where the magnetic induction, $\mathbf{B}$, and velocity, $\mathbf{U}$, vectors are obtained by averaging over the ensemble of turbulent pulsations, $\nu_{\mathrm{m}}$ is the regular magnetic diffusivity due to Ohmic dissipation, and $\beta$ is the eddy magnetic diffusivity (which can exceed the Ohmic diffusivity by several orders of magnitude).

An important role in producing the $\alpha$ effect is played by the helicity of the velocity field, $h = \mathbf{u} \cdot [\nabla \times \mathbf{u}]$, or ultimately, by its mean value $\langle h \rangle$ (here, we refer to averaging over the ensemble of random-field realizations and, in general, over the volume studied). We introduce the spectral helicity function $F(k, \omega)$ such that

$$\langle h \rangle = \iint F(k, \omega)\mathrm{d}k\mathrm{d}\omega \tag{4}$$

(here and below, $\mathbf{k}$ and $\omega$ will be the wave vector and frequency of a Fourier harmonic of any physical quantity considered); it can be shown that $\alpha$ also depends on this quantity. If $F(k, \omega) = 0$ (this condition is not

*E-mail: A.Getling@mail.ru

[1] As is often assumed in MHD studies, we consider it admissible to use the term *(magnetic) field* to mean *magnetic induction*, since the difference between these concepts is not significant in our case.

necessary, but is sufficient for the equality $\langle h \rangle = 0$ to hold), then $\alpha = 0$. Thus, the relationship between the mean velocity-field helicity and the $\alpha$ effect is fairly obvious. It is frequently claimed (or silently believed) that the $\alpha$ effect is only possible if $\langle h \rangle \neq 0$. As was shown by Gilbert et al. [6], a nonzero mean helicity is not necessary for the emergence of the $\alpha$ effect, although it favors its development.

Physically, the condition $\langle h \rangle \neq 0$ implies that local streams with right- and left-handed helicity (typically due to the action of the Coriolis force) have different numbers and/or intensities. In other words, the properties of the turbulent field $\mathbf{u}$ are not invariant with respect to the parity transformation (parity inversion)—transformation from a right-handed coordinate system $(x, y, z)$ to the left-handed system $(-x, -y, -z)$. Therefore, the velocity vectors for the pulsational fields $\mathbf{u}$ and $-\mathbf{u}$ at a given point and a given time are not equiprobable (so-called reflection non-invariance of the field).

The idea that reflection non-invariance of the turbulent field is of fundamental importance for the origin of the $\alpha$ effect is normally justified as follows. Since $[\nabla \times \mathbf{H}]$ is equal to the true vector of current density, $\mathbf{j}$, to within a numerical factor (which appears differently in different systems of units), both $\mathbf{H}$ and the magnetic induction $\mathbf{B}$ are pseudovectors. According to (1), $\mathcal{E}$ is a true vector, and it follows from (2) that the coefficient $\alpha$ should be a pseudoscalar, since it appears as a coefficient at a pseudovector in the expression for the true vector $\mathcal{E}$. Let us assume that the field $\mathbf{u}$ is reflection invariant. Then, $\alpha$ should not change with the parity transformation, since the ensemble of field realizations is not changed in this case. On the other hand, since $\alpha$ is a pseudoscalar, it should change its sign. Therefore, $\alpha = 0$ for reflection-invariant turbulence.

Helicity is also a pseudoscalar, since it can be obtained as a dot product of the true (polar) vector $\mathbf{u}$ and the axial vector (pseudovector) $[\nabla \times \mathbf{u}]$. Thus, a nonzero mean helicity guarantees that the turbulence is not reflection-invariant, which makes the $\alpha$ effect possible.

The magnitude of the $\alpha$ effect depends strongly on the properties of the turbulence; to estimate the $\alpha$ effect in a particular situation, some assumptions must be made about these properties. For the conditions in the solar convection zone, estimates of the coefficient $\alpha$ range from several cm/s to $10^4$ cm/s, which obviously implies a high degree of uncertainty in applying the results of calculations of dynamo models to actual solar conditions.

Note that representing the velocity field as an ensemble of chaotic pulsations is something of a stretch in the case of the solar convection zone. The convective flows of the solar plasma are considerably ordered. This can be seen not only from the existence of pronounced supergranulation cells, mesogranulation cells (the granulation is the subject of particular discussion [7, 8]), and giant cells (which have not yet been studied in detail [9]), but also from signs of more complex order in the structural organization of the flows [8, 10].

This study is aimed at demonstrating that a description of the solar dynamo in the language of mean-field electrodynamics can be based on more definite estimates of the helicity than those inferred from analyses of chaotic fields. Specifically, numerical simulations of cellular flows resembling the observed solar convection to some extent can be applied to directly calculate the mean helicity of the velocity field and, which is especially important, investigate how this helicity depends on the stratification parameters of the atmosphere (in this case, the helicity should obviously be averaged over the volume where the flow occurs rather than over an ensemble of realizations, since the velocity field is not random if the problem is formulated in this way).

Considering the helical component of the velocity field in calculations of the convective mechanism of magnetic-field amplification and structuring [11] should represent a step toward understanding the relationship between global and local solar magnetic fields.

In particular, the numerical calculations described here reveal the feature that the mean helicity of a cellular flow displays a maximum at a particular angular velocity of the medium, rather than varying monotonically. This fact can be explained in a natural way if we note that the rotation of the medium suppresses convection, and the limiting case of high rotation speeds corresponds to the conditions of the Taylor–Proudman theorem with a two-dimensional field of fluid velocity, which is constant along the direction of the angular-velocity vector of the medium.

## 2. FORMULATION OF THE PROBLEM AND ANALYSIS TECHNIQUE

As was already noted, non-zero helicity $h$ is mainly generated by the Coriolis force due to the rotation of the fluid as a whole. However, if the medium is fairly homogeneous (e.g., within the applicability of the Boussinesq approximation), the local helicity values for the convection field with opposite signs in different portions of the layer will cancel out and yield an average close to zero. This can easily be seen from the example of the velocity field obtained in the linear theory [12, Chapter 3] for Rayleigh–Bénard convection in a rotating layer, to which the Boussinesq approximation is applicable. Figure 1 shows that, for a hexagonal cell, if the angular rotational velocity of the layer $\mathbf{\Omega}$ is directed vertically, the trajectory of a fluid particle is swirled into a right-handed helix in the lower part and into a left-handed helix in the upper part of the layer. The homogeneity of the layer makes the pattern symmetric about the horizontal midplane, so that averaging the helicity over the layer yields $\langle h\rangle = 0$.

The mean helicity will be finite if there is considerable asymmetry between the upper and lower half-layers. Thus, it is reasonable to expect that, in contrast to the Boussinesq case, the mean helicity will differ from zero for the convection of a compressible gas. It is such convection that we will consider here.

Let us assume that a horizontal layer of a compressible fluid, $-0.5 \leq z/L \leq 0.5$ (where $L$ is the layer thickness and the $z$ axis is directed upward), rotating about the $z$ axis, is stratified polytropically: $p = K\rho^{\Gamma}$, $m = 1/(\Gamma - 1)$ ($p$ and $\rho$ are the pressure and density of the static atmosphere and $\Gamma$ and $m$ are the polytropic exponent and index, respectively). Then, using thermodynamical quantities calculated per unit mass of the material, we obtain the following unperturbed (static) distributions of the entropy $s$ and temperature $T$:

$$s = \frac{c_p}{\gamma}[1 - (\gamma - 1)m] \ln \frac{z - z_\infty}{z_{\text{ref}} - z_\infty}, \qquad (5)$$

$$T = \frac{c_s^2}{c_p(\gamma - 1)} \qquad (6)$$
$$= \frac{1}{c_p(m+1)} \frac{\gamma}{\gamma - 1} g_z (z - z_\infty)$$

(see, e.g., the manual for the *Pencil Code* software package [13], which we use here; our notation is in agreement with this manual). Here, $\gamma = c_p/c_v$ is the ratio of specific heats (the adiabatic exponent), $g_z =$ const $< 0$ is the gravitational acceleration, $c_{\text{s}} = \gamma p/\rho$ is the adiabatic sound speed, $z_\infty$ is the $z$ coordinate for which the gravitational potential in the form $\Phi = (z - z_\infty)(-g_z)$ vanishes, and $z_{\text{ref}}$ is the $z$ value for which the density and the sound speed assume some characteristic values $\rho_0$ and $c_{\text{s0}}$, chosen as reference values (we specified $s \equiv s_0 = 0$ at this height). The quantities $z_\infty$ and $z_{\text{ref}}$ are related as follows:

$$z_\infty = z_{\text{ref}} + (m+1) \frac{c_{\text{s0}}^2}{\gamma(-g_z)}. \qquad (7)$$

The temperature gradient in the absence of convective motions is

$$\beta = \frac{dT}{dz} = \frac{c_{\text{s}}^2}{c_p(\gamma - 1)} \qquad (8)$$

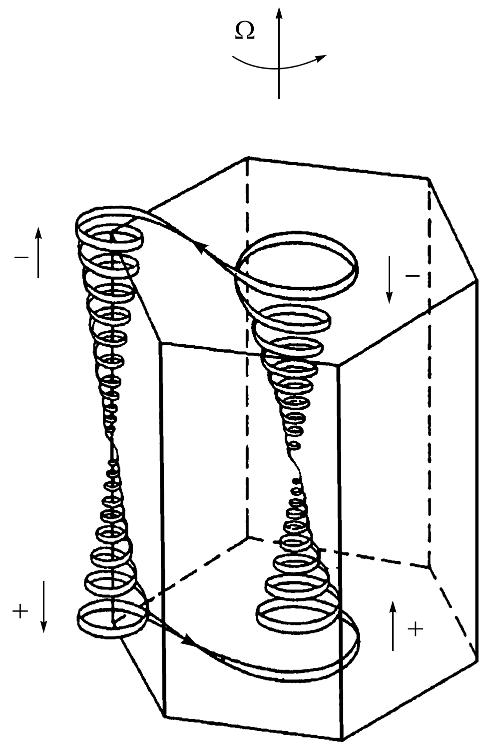


**Fig. 1.** Flow pattern in a hexagonal convection cell according to the linear theory [12]. The long vertical arrow at the top represents the rotational velocity of the entire fluid layer, $\mathbf{\Omega}$. The shorter arrows show the direction of the vorticity vector for different segments of the fluid particle trajectory; the signs of the local helicity (+ or −) are indicated near the arrows. Adapted from Fig. 25b of Chandrasekhar [12].

$$= \frac{1}{c_p(m+1)} \frac{\gamma}{\gamma - 1} g_z < 0.$$

It is known that the convective instability of a compressible medium is determined by the presence of a downward entropy gradient. For this reason, the definition of the Rayleigh number should be based on the entropy contrast (across the layer) rather than the temperature contrast [14]:

$$\text{Ra} = \frac{(-g_z)L^3}{\nu\chi} \frac{\Delta s}{c_p} \qquad (9)$$

($\nu$ is the kinematic viscosity of the medium and $\chi$ is its thermal diffusivity; $\Delta s$ is the difference of the static entropy values at the bottom and top boundaries). For polytropic stratification, this is equal to

$$\text{Ra} = \frac{(-g_z)L^3}{\nu\chi} \qquad (10)$$
$$\times \frac{1}{\gamma}[1 - (\gamma - 1)m] \ln \frac{-0.5 - z_\infty}{0.5 - z_\infty}.$$

We proceed from the Navier–Stokes system of equations for a compressible medium neglecting the

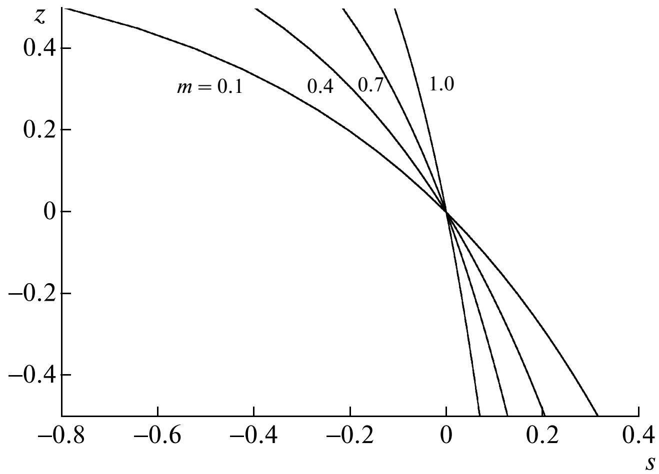


**Fig. 2.** Static $s(z)$ profiles for various $m$.

bulk viscosity and viscous dissipation of mechanical energy (we write the equations in the form that can be found in [13]).

As the boundary conditions, we assume (a) a no-slip condition at the upper and lower layer boundaries (considering them to be rigid),

$$\mathbf{v}\left(x, y, -\frac{1}{2}L\right) = \mathbf{v}\left(x, y, \frac{1}{2}L\right) = 0, \qquad (11)$$

(b) the isothermal condition at these same boundaries

$$T\left(x, y, -\frac{1}{2}L\right) = T_0 = \text{const}, \qquad (12)$$
$$T\left(x, y, \frac{1}{2}L\right) = T_1 = \text{const},$$

and (c) the condition that any physical quantity $f(x, y, z)$ be horizontally periodic,

$$f\left(-\frac{1}{2}L_x, y, z\right) = f\left(\frac{1}{2}L_x, y, z\right), \qquad (13)$$
$$f\left(x, -\frac{1}{2}L_y, z\right) = f\left(x, \frac{1}{2}L_y, z\right)$$

(where $L_x$ and $L_y$ are the spatial periods in the $x$ and $y$ coordinates).

We consider the medium to be motionless at the initial time and the entropy (or temperature) field to be weakly perturbed. We also specify a form for the initial perturbation that gives rise to a system of hexagonal cells at the initial development stage of the convection, with upflows in their central parts and downflows at the periphery. We choose the value $k = 2.22$ for the wave number of this perturbation, which corresponds to the critical wave number, $k_c$, for the classical Rayleigh–Bénard problem for the case of non-slip boundaries (note that the boundaries are assumed to be rigid in our calculations, so that $k_c = 3.12$).

We used a dimensionless representation of the variables with $L$, $-g_z$, and $\sqrt{L/(-g_z)}$ as the units of length, acceleration, and time, respectively. We obtained the unit of mass by setting the mean density equal to $\langle\rho\rangle = 1$. In addition, we assumed $c_{s0} = 1$ for the reference sound speed. The software package used internally sets $c_p = 1$. Thus, the units of entropy and temperature are defined by (5) and (6). (Note that the sound speed appears in the formula describing the stratification only as a thermodynamical quantity, and should not necessarily be measured in the same units as the flow speed; at the same time, since the entropy and temperature in the equations solved are related to the sound speed, their units must obviously be in agreement with the units for the other variables; however, we are not interested in determining the absolute entropy and temperature.)

The calculations described here were mainly carried out for a Rayleigh number of $\mathrm{Ra} = 20\,050.7$ and a Prandtl number of $\mathrm{Pr} = 1$; the polytropic index was varied in the range $m = 0.1$–$1$; for each $m$, we adjusted the $\nu$ and $\chi$ values so as to obtain the given Ra and Pr. The static entropy profiles $s(z)$ for various $m$ are shown in Fig. 2. For a given Ra, the dependence of the static temperature gradient $\beta$ on $m$ has the form shown in Fig. 3.

To simulate convective motions, we used the efficient and convenient *Pencil Code* software package developed by Brandenburg and Dobler [13]. This makes it possible to calculate flows of a compressible medium (in general, with the presence of a magnetic field) in finite differences of sixth order in the spatial coordinates and third order in time. The package can easily be adapted for a given number of available processor cores. The calculations for the problem at hand were executed using eight cores.

## 3. RESULTS AND DISCUSSION

The horizontal-periodicity conditions that we introduced for the physical quantities have a strong stabilizing effect on the flow [15, Chapter 6]. For this reason, hexagonal cells develop from the weak initial thermal perturbations in a typical calculated evolutionary scenario, then persist over a relatively long period (Fig. 4); the velocity amplitude (the maximum value $u_{\max}$) and the mean helicity vary slowly in this quasi-stationary regime (Fig. 5). Let us note that the velocity field in the cells exhibits a fairly rapid change to a "two-vortex" structure [16] during its early evolution, viz., a downflow develops in the central part of the cell (as at its periphery), being surrounded by an annular upflow region; the flow thus forms two vortices in a meridional cross section of the cell (which was the basis for denoting this a two-vortex structure). Precisely this structure is demonstrated by the cells during the period of their quasi-stationary behavior. As can be seen from numerous situations studied previously for non-rotating layers [16], this is an effect of the non-optimal horizontal flow scale enforced by the initial conditions: the $x$ and $y$ sizes of the cell considerably exceed the optimal scale, and the flow adjusts itself to a smaller scale (the issue of scale optimum for two-dimensional convection in a non-rotating layer is extensively discussed in [15, Chapter 6]). The specific appearance of the cells is indicative of flow swirling qualitatively similar to that shown in Fig. 1, but asymmetric about the horizontal midplane of the layer. Eventually, the cellular pattern breaks down and changes into a system of rolls or a more complex, disordered pattern.

Although the computations were mostly done for a domain with horizontal sizes equal to the fundamental periods of the initial perturbation in $x$ and $y$, some checking runs were carried out for domains of double or triple this size in each horizontal coordinate, with the spatial grid steps preserved. For the parameter space explored, doubling or tripling the horizontal size of the domain has little effect on both the flow stability and the values and time variation of the maximum velocity and mean helicity.

Let us consider the dependence of the velocity amplitude and mean helicity in the stage of quasi-stationary cells on the stratification and rotational speed of the layer (Fig. 6). With decreasing $m$ and, accordingly, increasing curvature of the static entropy profile (Fig. 2), i.e., with growing asymmetry between the upper and lower halves of the layer, the height distribution $h(z)$ for given $x$ and $y$ becomes progressively less symmetric. Accordingly, the helicity averaged

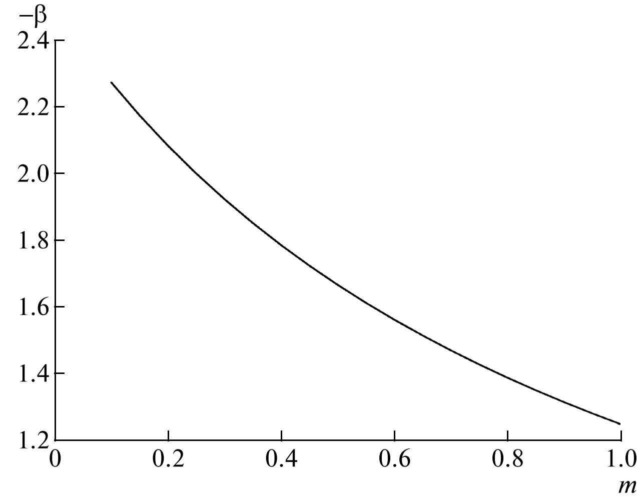


**Fig. 3.** The dependence $\beta(m)$ for Ra = 20 050.6.

over the layer, $\langle h\rangle$, reaches larger values, so that its $\Omega$ dependence has a pronounced maximum. This is due to the fact that, as $\Omega$ grows, the conditions approach those for the Taylor–Proudman theorem, ultimately quenching the convection. However, as can be seen from Fig. 6, the suppression of the helicity preceeds the quenching of the convection.

Note that, in his studies of angular-momentum transfer by convective turbulence in a rotating compressible atmosphere, Kichatinov [17] did not find an explicit decrease in the mean flow helicity for large $\Omega$. According to his formula, the mean helicity $\langle h\rangle$ approaches a constant value as $\Omega \to \infty$. However, since the radial gradient of the rms turbulent velocity appears as a factor in this formula, and rapid rotation quenches convection, reducing this velocity, there should nevertheless be a decrease in $\langle h\rangle$ for large $\Omega$.

Thus, for a given stratification, there exists a certain optimum rotational velocity for the generation of flow helicity. As an example, let us derive an estimate for some conditions in the range of quantities that can be expected for the Sun. If we assume $\Omega_{z\odot} \sim 2\times10^{-6}$ rad/s and a gravitational acceleration of $g_\odot \sim 260$ m/s, then, for a characteristic layer thickness of $L \sim 60$ Mm, we find the time unit used in our calculations to be $t_0 = \sqrt{L/g_\odot} \sim 480$ s, and, accordingly, the dimensionless angular velocity to be $\Omega \sim 10^{-3}$. Modern models of the internal structure of the solar convection zone (see, e.g., [18]) yield for the depth interval $0.1{-}0.2R_\odot$ a conventional (since the polytropic stratification is an idealized case) value of $m \sim 1$. Thus, the point in Fig. 6b that corresponds to this velocity can probably not be far from the helicity maximum for $m = 1$. However, to directly estimate

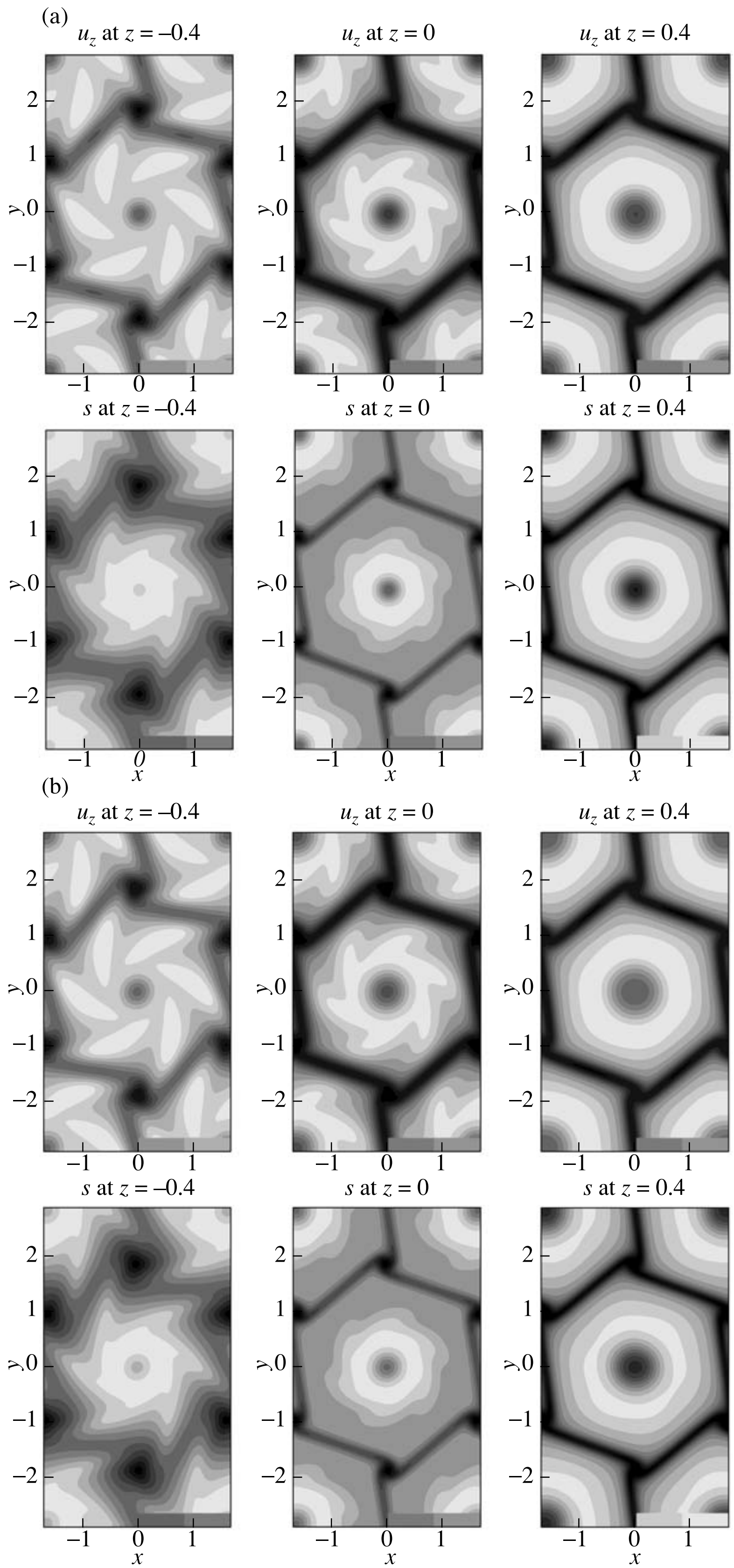


**Fig. 4.** Distributions of the vertical velocity component (upper rows) and the entropy perturbation (lower rows) over three horizontal cross sections of a convection cell during the steady-state regime at (a) $t = 40$ and (b) $t = 80$ in the run for $\Omega = 0.04$, $m = 0.1$. Zero values of quantities correspond to the transition between the gradations of gray shown in the left and right halves of the narrow horizontal strip at the lower right corner of each diagram.

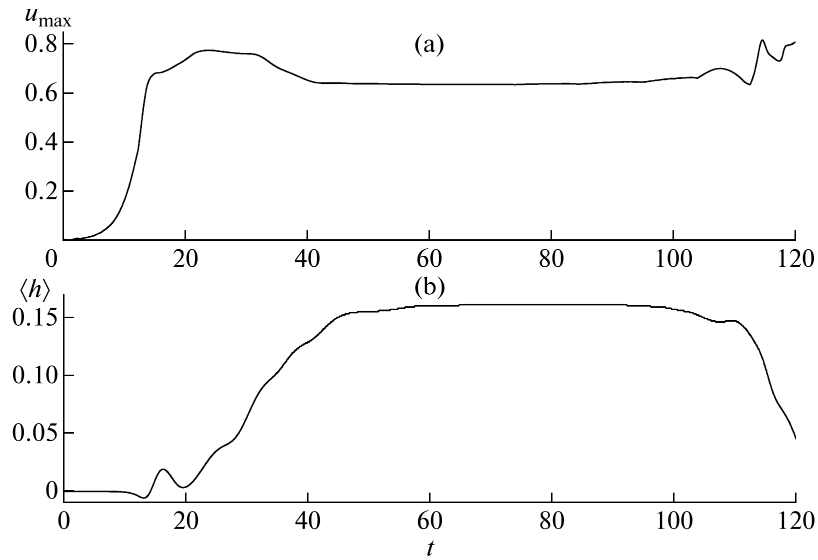


**Fig. 5.** Time variations of the (a) maximum velocity in the cell, $u_{max}$, and (b) mean helicity $\langle h \rangle$ in the run for $\Omega = 0.04$, $m = 0.1$.

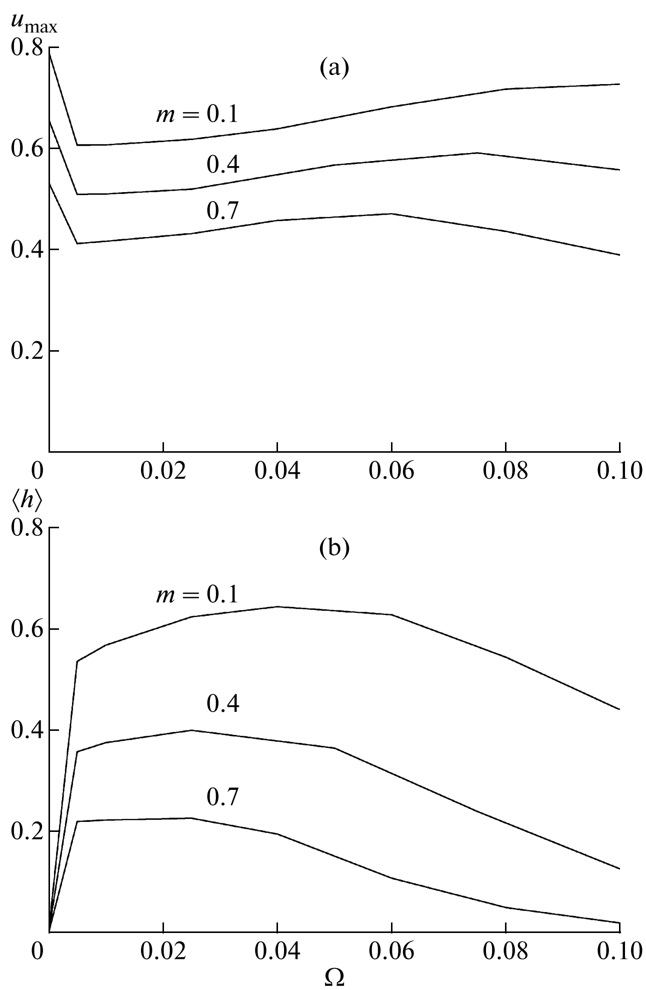


**Fig. 6.** (a) Maximum velocity in the cell, $u_{max}$, and (b) mean helicity, $\langle h \rangle$, in the steady-convection regime as functions of the angular speed $\Omega$.

the $\alpha$ effect for these conditions, we need more complete information on the relationship between this effect and helicity than is currently available.

In general, such "deterministic" investigations of the helicities of quasi-regular convective flows should offer possibilities for considerably reducing the uncertainty in estimates used in the theory of mean-field dynamos.

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